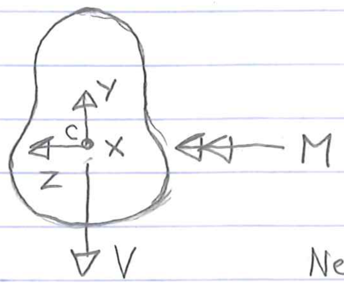
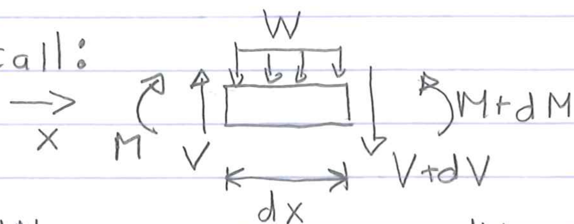
31. Deflections of beams

Linear elastic, homogeneous, isotropic material. Planar bending of symmetric sections.



x-y is plane of symmetry.
x and y are principal, centroidal axes.
z is neutral axis.
$I_z = I$, constant EI
Neglect contribution of V to deflections.

Recall:



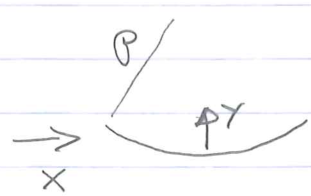
w is the distributed load on the beam (force per length; can vary with x)

$$\frac{dM}{dx} = V \quad (1) \qquad\qquad \frac{dV}{dx} = -w \quad (2)$$

Recall: $\frac{1}{\rho} = \frac{M}{EI}$ (3) where ρ is the radius of curvature of the deflected neutral axis.

We are interested in the vertical deflection of the beam. The text book calls this deflection y (positive upwards). y is actually the vertical deflection of the centroid. Other points have slightly different vertical deflections due to the Poisson effect.

Recall from calculus: $\frac{1}{\rho} = \frac{d^2y}{dx^2}$ (4) for small curvature.



From (3) and (4): $\frac{d^2y}{dx^2} = \frac{M}{EI}$ (5)

From (1) and (5): $\frac{d^3y}{dx^3} = \frac{V}{EI}$ (6)

From (2) and (6): $$\frac{d^4y}{dx^4} = -\frac{W}{EI} \qquad (7)$$

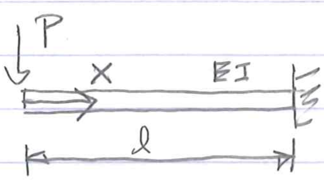
Equation (7) is the beam differential equation from which the deflection y is computed. W and EI are known.

The general solution to (7) is

$$y = a + bx + cx^2 + dx^3 + \text{the particular solution} \qquad (8)$$

where a, b, c, d are determined by the boundary conditions, and the particular solution depends on W.

Example:



$$Y = a + bx + cx^2 + dx^3$$ (particular solution is zero; P is incorporated into the shear boundary condition at $x=0$)

$Y(x=l)=0$: $\quad a + bl + cl^2 + dl^3 = 0$

$\frac{dy}{dx}(x=l) = 0$: $\quad b + 2cl + 3dl^2 = 0$

$M(x=0) = 0$: $\quad EI \left.\frac{d^2y}{dx^2}\right|_{x=0} = 0 \Rightarrow 2c = 0$

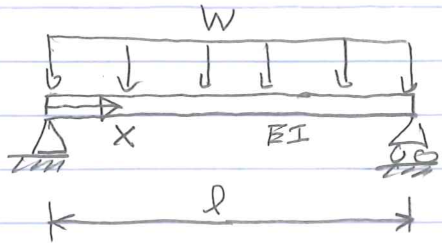
$V(x>0) = -P$: $\quad EI \left.\frac{d^3y}{dx^3}\right|_{x>0} = -P \Rightarrow EI \cdot 6d = 0$

Solving for a,b,c,d gives $\quad a = \frac{-Pl^3}{3EI}$, $b = \frac{Pl^2}{2EI}$, $c = 0$, $d = \frac{-P}{6EI}$

Thus,
$$y = \frac{P}{EI}\left(-\frac{l^3}{3} + \frac{l^2}{2}x - \frac{1}{6}x^3\right)$$

and
$$y_{max} = -\frac{Pl^3}{3EI} \quad \text{at } x = 0.$$

Example:



(W is constant)

$$y = a + bx + cx^2 + dx^3 - \frac{W}{24EI} x^4$$

$y(x=0) = 0$: $a = 0$

$y(x=\ell) = 0$: $a + b\ell + c\ell^2 + d\ell^3 - \frac{W}{24EI}\ell^4 = 0$

$M(x=0) = 0$: $EI \left.\frac{d^2y}{dx^2}\right|_{x=0} = 0 \Rightarrow 2c = 0$

$M(x=\ell) = 0$: $EI \left.\frac{d^2y}{dx^2}\right|_{x=\ell} = 0 \Rightarrow (2c + 6d\ell)EI - \frac{W}{2}\ell^2 = 0$

Solving for a, b, c, d gives $a = 0$, $b = -\frac{W\ell^3}{24EI}$, $c = 0$, $d = \frac{W\ell}{12EI}$

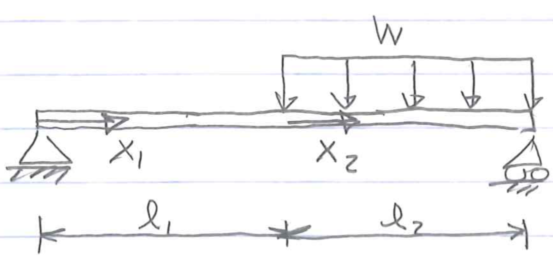
Thus,

$$y = \frac{W}{24EI}\left(-x\ell^3 + 2\ell x^3 - x^4\right)$$

and

$$y_{max} = -\frac{5}{384}\frac{W\ell^4}{EI} \quad \text{at } x = \frac{\ell}{2}$$

Example:



Divide the beam into pieces 1 and 2.

For $0 \le x_1 \le \ell_1$ $\quad y_1 = a_1 + b_1 x_1 + c_1 x_1^2 + d_1 x_1^3$

$0 \le x_2 \le \ell_2$ $\quad y_2 = a_2 + b_2 x_2 + c_2 x_2^2 + d_2 x_2^3 - \frac{W}{24EI} x_2^4$

We need 8 equations to determine the 8 unknown constants.

Boundary conditions: $Y_1(X_1=0) = 0$ $\quad M_1(X_1=0) = 0$

$Y_2(X_2=\ell_2) = 0$ $\quad M_2(X_2=\ell_2) = 0$

Matching conditions: $Y_1(X_1=\ell_1) = Y_2(X_2=0)$

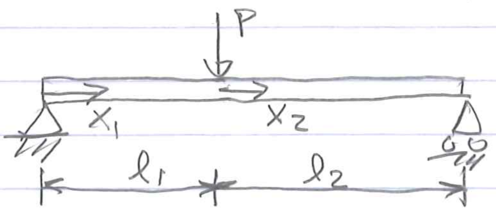
$dY_1/dx_1 \,|_{x_1=\ell_1} = dY_2/dX_2 \,|_{X_2=0}$

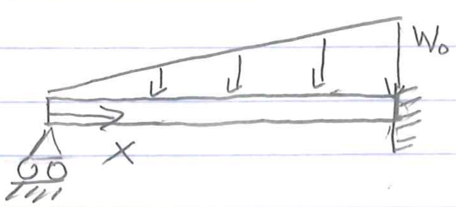
$M_1(X_1=\ell_1) = M_2(X_2=0)$

$V_1(X_1=\ell_1) = V_2(X_2=0)$

The solution is algebraically complicated and won't be completed here.

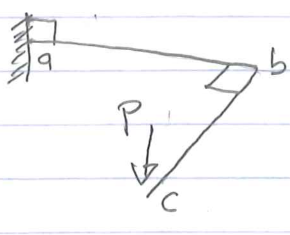
Other examples:



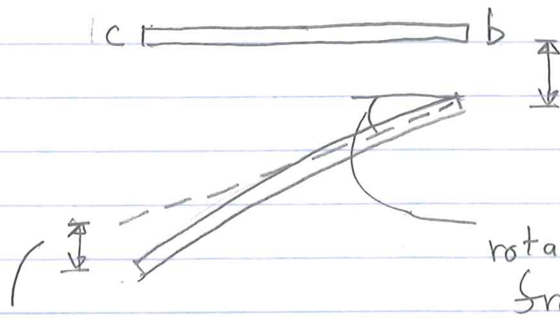


This one is statically indeterminate.

Example:



Beams a-b and b-c are in a horizontal plane. P is downward.



displacement of beam a-b from end load P

rotation due to twisting of beam a-b from torque $P \cdot \ell_{bc}$.

displacement of beam b-c from end load P